Math Modeling, Week 6

Decision making

Data: Response probability, response time (RT)

Manipulations: Task difficulty, speed-accuracy tradeoff

Signal-detection theory

Binary hypothesis set $\{H_1, H_2\}$

Univariate observation x

Likelihoods $p_i(x) = \mathrm{p}(x|H_i)$

Posterior $\frac{\Pr[H_1|x]}{\Pr[H_2|x]} = \frac{\Pr[H_1]}{\Pr[H_2]} \cdot \frac{p_1(x)}{p_2(x)}$

Optimal response: 1 if $\frac{p_1(x)}{p_2(x)} > \frac{\Pr[H_2]}{\Pr[H_1]}$, else 2

Can also include asymmetric rewards: response 1 iff $\frac{p_1(x)}{p_2(x)} > \frac{\Pr[H_2]}{\Pr[H_1]} \cdot \frac{R_{2|H_2} - R_{1|H_2}}{R_{1|H_1} - R_{2|H_1}}$

Models response bias

Random walk model

Multiple observations $(x_i)_{i=1,2,\ldots}$

Posterior log-odds: $\ln \frac{\Pr[H_1|\mathbf{x}]}{\Pr[H_2|\mathbf{x}]} = \ln \frac{\Pr[H_1]}{\Pr[H_2]} + \sum_i \ln \frac{p_1(x_i)}{p_2(x_i)}$

Evidence: Belief on log-odds scale

Prior log-odds, posterior log-odds, loglikelihood ratio

Simply update: $E_n = E_0 + \sum_{i=1}^{n} L(x_i)$

Random walk

Stopping rule

After trial n, sample more or stop and give a response

RT $= n\tau$ for some time increment τ

Optimal: continue until $|E_n| > \alpha$ (decision threshold)

Wald-Wolfowitz theorem: minimizes mean RT for a given Pr[correct]

Speed-accuracy tradeoff

Higher threshold $\rightarrow$ longer mean RT but higher probability correct

Task difficulty

Average rate of random walk

Separation of p_1 and p_2

$$\begin{aligned} \mathrm{E}[L(x)|H_1] &= \int \ln \frac{p_1(x)}{p_2(x)} \cdot p_1(x)\,\mathrm{d}x \\ &= KL(p_1||p_2) \end{aligned}$$

$$\mathrm{E}[L(x)|H_2] = -KL(p_2||p_1)$$

Kullback-Leibler divergence: difference between two distributions, always nonnegative

Diffusion model

Drop rational interpretation

Process model: accumulation of psychological evidence

Continuous time

Limit of timestep $\tau \rightarrow 0$

Evidence rate $\mathrm{E}[\Delta E]/\tau \rightarrow \mu$

Variance $\mathrm{var}[\Delta E]/\tau \rightarrow \sigma^2$

Wiener diffusion process (directed Brownian motion)

Continuous-time stochastic process

Markov: $E(t_2) - E(t_1) \perp E(t_1)$ for $t_1 < t_2$

Linear growth: $\mathrm{E}[E(t_2) - E(t_1)] = (t_2 - t_1)\cdot\mu$

Linear noise: $\mathrm{var}[E(t_2) - E(t_1)] = (t_2 - t_1)\cdot\sigma^2$

Central limit theorem: $E(t_2) - E(t_1) \sim \mathcal{N}((t_2 - t_1)\cdot\mu, (t_2 - t_1)\cdot\sigma^2)$

Parameters

- Starting point $z = E(0)$: response bias
- Drift rate μ: task difficulty
- Thresholds $\{0, a\}$: speed-accuracy tradeoff
- Diffusion rate σ^2: noise, usually ignored by scaling other parameters
    - Predictions unchanged under $(z, \mu, a, \sigma) \rightarrow (z \cdot c, \mu \cdot c, a \cdot c, \sigma \cdot c)$ for any c

Predictions

- Simulation of each trial, $\tau \approx 1$ ms
- Analytic series for density

$$\mathrm{p}[r_1, t | H_1] = \frac{\pi\sigma^2}{a^2} e^{\frac{\mu(a-z)}{\sigma^2}} \sum_{k=1}^{\infty} k \sin\left(\frac{k\pi(a-z)}{a}\right) e^{-\frac{k^2\pi^2\sigma^4 + a^2\mu^2}{2a^2\sigma^2}t}$$

- or for cumulative distribution

$$\mathrm{p}[r_1, RT \leq t | H_1] = \mathrm{p}[r_1 | H_1] - \frac{\pi\sigma^2}{a^2} e^{\frac{\mu(a-z)}{\sigma^2}} \sum_{k=1}^{\infty} \frac{2ka^2\sigma^2}{k^2\pi^2\sigma^4 + a^2\mu^2} \sin\left(\frac{k\pi(a-z)}{a}\right) e^{-\frac{k^2\pi^2\sigma^4 + a^2\mu^2}{2a^2\sigma^2}t}$$

- Marginal response probability

$$\mathrm{p}[r_1 | H_1] = \frac{1 - e^{-\frac{2\mu z}{\sigma^2}}}{1 - e^{-\frac{2\mu a}{\sigma^2}}}$$

Fitting by χ^2 for discrete probabilities

- Bin RT by empirical quantiles, usually .1, .3, .5, .7, .9
- 12 possible outcomes for each trial

$$\chi^2 = \sum_{\text{bin}} \frac{\left(f^{\text{model}} - f^{\text{data}}\right)^2}{f^{\text{model}}}$$